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Estimation of plate elastic moduli through vibration testing ☆

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Abstract

This paper considers the identification problem for 2D-structures by comparing a modal method with a new method based on the estimation of the dispersion equation in k-space. Both methods are validated by numerical simulation and by measurements based on an acoustic holography experiment.

Keywords: Identification; Near-field acoustic holography; Dispersion equation; Elastic constants

1. Introduction

This paper is concerned with the general problem of acoustic comfort in air transportation with a special emphasis on aircraft and helicopter cabins. The results presented in this paper address both the diagnosis and modelling problems. Diagnosis is achieved by near-field acoustic holography techniques (NAH for short). NAH has been used to measure the vibro-acoustic fields (sound pressure, vibration velocity, sound intensity) on all the inner panels of a helicopter cabin. The experiment and the measure collection process

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are described in [1]. Our main interest in NAH here is to provide a measurement tool giving access to the 2D-vibration field without contact with these panels. Moreover, it is important to stress that one of the other advantages of NAH as compared to measurement techniques such as optical interferometry or laser vibrometry, is its low sensibility with respect to noisy data, and this feature is crucial for helicopter in-flight measurements.

In the present paper, the structures so obtained are used to model the vibrations and the acoustic radiations of the different cabin panels. Modelling cabin noise requires the identification of the mechanical properties of the radiating structures. The latter are generally made of stiffened composite panels, and the measurement of their elasticity constant is rather delicate. Therefore, we searched for an equivalent model based on the definition of an equivalent orthotropic plate for these complex structures, and we used "in situ" measurements to achieve this goal. To be specific, we developed this strategy from measurements made on a particular set of plates. However, the reader should easily convince herself that no significant loss of generality ensues, as the methods presented here can be applied to more complex cases. In fact, our objective was not to solve the direct plate vibration problem. Our main motivation was to use the plate vibration equations to validate the methods developed here, shooting for an estimate of the plate elastic moduli.

At low frequencies, the behavior of these structures is mostly modal driven: the vibrations can be described by the sum of the mode shapes of the structure. In this regime, a modal identification approach is proposed. At medium or high frequencies where the modal density becomes very important, this modal approach becomes inefficient. We provide a propagative description of the dynamics Wave propagation parameters are identified, most notably the elastic wave numbers appearing in the dispersion equation.

Modelling the dynamic response of an orthotropic plate is a challenging problem which has been widely studied in the recent past. One of the first method used to tackle this problem was proposed by Aksu and Ali [2]. The authors used a finite difference method in order to compute the vibration response of a stiffened plate. In the same vein, Mukhopadhyay [3] proposed a different finite difference method to solve the vibration problem of an orthotropic plate. More recently, the finite element method was used to compute the dynamic response of orthotropic plates in [4,5]. A semi-analytical method was proposed by Dalaei and Kerr [6] in 1996 to compute the free vibrations of an orthotropic plate: the authors used an expansion of the solution in harmonics, recasting their approach in a Galerkin-like approach.

All the *direct problems* require the knowledge of the elastic constants of the orthotropic panel under investigation. In this sense, a given orthotropic panel is characterized by a set of four independent elastic constants [7]. Standard tests are available to determine these constants through mechanical probes [8]. However, these measurements are more delicate than for isotropic plates. As an alternative solution to the direct problem attempting to resolve the vibrations of an orthotropic plate, some authors have proposed *inverse methods* to identify the elastic constants of an orthotropic panel through vibration testing. Three approaches have been proposed. The first approach is based on measurements of the natural frequencies of the tested panel. An inverse identification technique is then used to obtain converged values of the elastic constants from the measured natural frequencies. This method requires the direct problem generally by Rayleigh–Ritz or Galerkin techniques is presented in [7,9] for example. The second approach presented by Grédiac and Paris [10], uses the same data that the first one, namely the measured natural frequencies of the panel, but works directly: neither initial values of the mechanical properties of the

plate, nor iterative computation are required. Nevertheless, these two methods are low frequency methods. The third method proposed by Berthaut and Ichchou [11] works at high frequencies. It is also a direct method in which the authors proposed to identify the parameters of the dispersion equation from vibrational velocity maps of the plate, measured by a laser vibro-meter, through projection of this measure on an inhomogeneous waves' base.

The method we propose in the present paper is direct, so it can be compared to [10]. It performs well at low frequencies. Also, it relies on the identification of a set of triplets (m,n,ω_{mn}) which represent the mode shapes W_{mn} and the natural frequencies of the structure. When compared to the approach used in [10], the main advantage of our method is its simplicity. Indeed [10] requires the computation of the variational principle for the measured mode shapes, forcing an integration over the measured mode shapes W_{mn}. In our approach, one merely needs to identify the triplets (m,n,ω_{mn}) corresponding to the mode shapes W_{mn} which represents a significant computational saving.

The paper is organized as follows: first a short preview of the main characteristics of the vibro-acoustic response of a specially orthotropic plate is given. The related equations are used in the numerical validation step presented later on. Then the principle of the identification of the elasticity constants of a special orthotropic plate is presented. It is based on different boundary conditions, such as simply supported and a clamped plates. In the following section, we validate this approach with the numerical analysis of a carbon–epoxy composite plate. Then, the dispersion equation of the identification is presented and compared to the modal approach according to the numerical model. Finally, we apply these methods to an experimental case study using near-field acoustic holography measures. We conclude with some perspectives for future work.

2. Vibro-acoustic response of a specially orthotropic panel

We use cartesian coordinates $(0,\mathbf{x},\mathbf{y},\mathbf{z})$ in *free space* $\mathbb{R}^3$. A plane elastic rectangular panel Σ with length a, width b, and with a constant thickness h, is submitted to two different boundary conditions: simply supported boundary, or clamped boundary onto its boundary $\partial\Sigma$. We are interested in an industrial panel as one can find in an helicopter cabin. Since these panels are fairly rigid, we assume that the influence of the fluid on the vibro-acoustic response of the panel can be neglected. It is also assumed that the damping phenomenon can be neglected: this is a valid approximation since we consider stiffened steel panels or advanced composites such as carbon fiber-reinforced plastics.

In this section, we derive the time-harmonic system of equations governing the transverse vibrations of an "in vacuo" thin, specially orthotropic elastic plate.

Let us denote $u(M)$ the transverse displacement of the plate Σ at the point M with coordinates (x,y), submitted to an external time-harmonic force $f(M)$. The functions u and f are solutions of the following system of equations:

$$\begin{cases} D_1 \frac{\partial^4 U}{\partial x^4} + D_2 \frac{\partial^4 U}{\partial y^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 U}{\partial x^2 \partial y^2} - m\omega^2 U = -F \quad \forall M \in \Sigma \quad \text{(a)} \\ \text{Boundary conditions for } U \text{ onto } \partial\Sigma \quad \text{(b)} \end{cases} \tag{1}$$

where m denotes the mass of the plate per unit of surface, D_1, D_2 and D_{12} are the bending stiffness factors, D_{66} is the torsional stiffness which depends upon the transverse Young's moduli E_1 and E_2, the shear-modulus G_{12} and the major and minor Poisson ratios ν_{12} and ν_{21} as explained for example, in [5,7,12]:

$$D_1 = \frac{E_1 h^3}{12(1 - \nu_{12}\nu_{21})}, \quad D_2 = \frac{E_2 h^3}{12(1 - \nu_{12}\nu_{21})} \tag{2}$$

In the sequel, we denote by D_3 the following real number:

$$D_3 = D_{12} + 2D_{66} = \nu_{12} D_2 + \frac{G_{12} h^3}{6}, \quad \nu_{12} E_2 = \nu_{21} E_1 \tag{3}$$

Two boundary conditions (1b) are considered in the present paper:

Clamped plate: $u = 0$ if $M \in \partial\Sigma$, $\partial_x u = 0$ if $x \in \{0,a\}$ and $\partial_y u = 0$ if $y \in \{0,b\}$.
Simply supported plate: $u = 0$ if $M \in \partial\Sigma$, $\partial^2_{x^2} u = 0$ if $x \in \{0,a\}$ and $\partial^2_{y^2} u = 0$ if $y \in \{0,b\}$.

The identification method we propose here does not need to explicitly solve the above system of equations (1a,1b) for the displacement of the plate and its boundary conditions. This system is solved only for validation purposes: the numerical solutions of (1a,1b) for the bending stiffness of a given orthotropic plate D_1, D_2 and D_{12} and the torsional stiffness D_{66}, will give the vibrating responses of well-known specially orthotropic panels. These responses will then be introduced in our identification procedure. Finally, the identified bending stiffness factors will be compared to the computed ones.

The system (1a,1b) is solved numerically by using a high-order finite difference method to compute the bi-harmonic operator appearing in the plate equation (1a). The numerical study of the method used to solve the system (1a,1b) is presented in [13].

In the sequel, we examine the possibility of determining the elastic constants of a thin specially orthotropic plate by means of vibration testing and modal analysis.

3. Evaluation through low frequency modal analysis of elastic constants of a specially orthotropic plate

In this section a method is proposed to evaluate the transverse Young moduli E_1, E_2 and the in-plane shear modulus G_{12} of a specially orthotropic plate by vibration testing and modal analysis of a plate submitted to two different boundary conditions: simply supported boundary and clamped boundary.

3.1. Identification on a simply supported plate

Consider a specially orthotropic simply supported rectangular plate of dimensions $[a \times b]$. Let us denote by W_{mn} the eigen-modes and Ω_{mn} the eigen-angular frequencies of this plate. The functions W_{mn} and Ω_{mn} solve the following system of equations:

$$\begin{cases} D_1 \frac{\partial^4 W_{mn}}{\partial x^4} + D_2 \frac{\partial^4 W_{mn}}{\partial y^4} + 2D_3 \frac{\partial^4 W_{mn}}{\partial x^2 \partial y^2} = m\Omega^2_{mn} W_{mn} & \text{(a)} \\ W_{mn} = 0 \quad \text{if } x \in \{0,a\} \quad \text{or} \quad \text{if } y \in \{0,b\} & \text{(b)} \\ \partial^2_{x^2} W_{mn} = 0 \quad \text{if } x \in \{0,a\} & \text{(c)} \\ \partial^2_{y^2} W_{mn} = 0 \quad \text{if } y \in \{0,b\} & \text{(d)} \end{cases} \tag{4}$$

The eigen-modes W_{mn} and the eigen-angular frequencies Ω_{mn} of a simply supported specially orthotropic plate of dimension $[a \times b]$ are given by (see e.g. [12]):

$$W_{mn}(x,y) = \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right) \tag{5}$$

Let us denote by ξ and η the dual variables of x and y in the Fourier transform domain, as defined by:

$$\hat{W}_{mn}(\xi,\eta) = \int\int_{\mathbb{R}^2} W_{mn}(x,y)\mathrm{e}^{\mathrm{i}\xi x}\mathrm{e}^{\mathrm{i}\eta y}\,\mathrm{d}x\,\mathrm{d}y \tag{6}$$

The variables ξ and η are the wave numbers in the directions x and y. Taking the Fourier transform in space of (4a) leads to:

$$D_1\xi^4 + D_2\eta^4 + 2D_3\xi^2\eta^2 = m\Omega_{mn}^2 \tag{7}$$

When the space Fourier transform (6) is performed in an analytical way, then the expressions of the wave numbers ξ and η associated to the mode shape W_{mn} defined by the formula (5) become:

$$\xi = \xi_m = \frac{m\pi}{a}, \quad \eta = \eta_n = \frac{n\pi}{b} \tag{8}$$

The above expressions of ξ_m and η_n are introduced in Eq. (7) satisfied by the eigen-mode $(\tilde{W}_{mn}, \Omega_{mn})$. This last equation when written in matrix form, gives:

$$\left[\xi_i^4, \eta_j^4, 2\xi_i^2\eta_j^2\right]\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = m\Omega_{ij}^2 \tag{9}$$

Suppose now that N couples of eigen-modes/eigen-frequencies of the simply-supported plate are known

$$\underbrace{\left\{(W_{ij},\Omega_{ij}),\ldots,(W_{kl},\Omega_{kl})\right\}}_{N\text{ terms}} \tag{10}$$

The equations satisfied by each couple (W_{ij},Ω_{ij}) define a set of N equations, with 3 unknowns: The flexural rigidities D_1, D_2, and D_3:

$$\underbrace{\begin{bmatrix} \xi_i^4, \eta_j^4, 2\xi_i^2\eta_j^2 \\ \cdot \\ \cdot \\ \cdot \\ \xi_k^4, \eta_l^4, 2\xi_k^2\eta_l^2 \end{bmatrix}}_{[A]} \cdot \underbrace{\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix}}_{[D]} = \underbrace{\begin{bmatrix} m\Omega_{ij}^2 \\ \cdot \\ \cdot \\ \cdot \\ m\Omega_{kl}^2 \end{bmatrix}}_{[B]} \Bigg\} N \text{ lines} \tag{11}$$

When N is greater than 3, the system (11) can be solved by a least squares method. Let $[A]^{\mathrm{t}}$ denote the transposed matrix of the matrix $[A]$, and $\{[A]^{\mathrm{t}}\cdot[A]\}^{-1}$ the inverse of the matrix $\{[A]^{\mathrm{t}}\cdot[A]\}$. The solution of the system (11) is given by:

$$[D] = \left\{[A]^{\mathrm{t}}\cdot[A]\right\}^{-1}[A]^{\mathrm{t}}\cdot[B] \tag{12}$$

The bending stiffness factor is given by the resolution of (12), and the Young and shear moduli of the plate come from Eqs. (2) and (3):

$$
\begin{aligned}
E_1 &= \frac{12}{h^3}\left(D_1 - v_{12}^2 D_2\right)\\
E_2 &= \frac{D_2}{D_1}\frac{12}{h^3}\left(D_1 - v_{12}^2 D_2\right)\\
G_{12} &= \frac{6}{h^3}\left(D_3 - v_{12} D_2\right)\\
v_{21} &= \frac{v_{12} D_2}{D_1}
\end{aligned}
\tag{13}
$$

3.2. Identification on a clamped plate

The identification of the Young and the shear moduli of a clamped plate uses the results presented in [12,14]. For a clamped plate, the typical mode shapes are written as follows:

$$W_{mn}(x,y) = X_m(x) Y_n(y) \tag{14}$$

where $X_m(x)$ and $Y_n(y)$ are the mode shapes of the associated clamped beams of respective lengths a and b, so that:

$$X_m(x) = A_{1m}\sin(\alpha_m x) + A_{2m}\cos(\alpha_m x) + A_{3m}\sinh(\alpha_m x) + A_{4m}\cosh(\alpha_m x) \tag{15}$$

$$Y_n(y) = B_{1n}\sin(\beta_n y) + B_{2n}\cos(\beta_n y) + B_{3n}\sinh(\beta_n y) + B_{4n}\cosh(\beta_n y) \tag{16}$$

where the parameters A_{1m}, A_{2m}, A_{3m}, A_{4m}, B_{1n}, B_{2n}, B_{3n}, B_{4n}, α_m and β_n proposed by [12] are:

$$
\begin{aligned}
&A_{1m} = -A_{3m} = 1, \quad A_{2m} = -A_{4m} = \frac{\sinh(\alpha_m a) - \sin(\alpha_m a)}{\cos(\alpha_m a) - \cosh(\alpha_m a)},\\
&B_{1n} = -B_{3n} = 1, \quad B_{2n} = -B_{4n} = \frac{\sinh(\beta_n b) - \sin(\beta_n b)}{\cos(\beta_n b) - \cosh(\beta_n b)},\\
&\alpha_m = \lambda_m / a, \quad \beta_n = \lambda_n / b \qquad (m = 1,2,\ldots;\ n = 1,2,\ldots)
\end{aligned}
\tag{17}
$$

The parameters λ_m and λ_n in the two directions are obtained by solving the following equation:

$$\cosh(\lambda)\cos(\lambda) - 1 = 0$$

The 5 first zeros of the previous equation are calculated numerically by a dichotomic algorithm. The solutions are:

$$
\begin{aligned}
\lambda_1 &= 4.73004074692726\\
\lambda_2 &= 7.85320462584495\\
\lambda_3 &= 10.99560784101484\\
\lambda_4 &= 14.13716549277302\\
\lambda_5 &= 17.27875965833661
\end{aligned}
\tag{18}
$$

and when $m,n > 5$ the parameters λ_i $(i = m,n)$ are accurately approximated by the formula:

$$\lambda_i = \frac{\pi}{2} + i\pi \quad \text{if } i > 5.$$

Then, the wave numbers ξ_m and η_n come from the Fourier transform in the space domain (6) applied to the mode shape (14):

$$\xi_m = \frac{\alpha_m}{a}, \quad \eta_n = \frac{\alpha_n}{b}$$

When $m,n > 5$, the wave numbers in the directions x and y, are determined analytically

$$\xi_m = \left(m + \frac{1}{2}\right)\frac{\pi}{a}, \quad \eta_n = \left(n + \frac{1}{2}\right)\frac{\pi}{b} \tag{19}$$

The identification of the bending rigidities of a clamped plate begins with the modal analysis of the structure and the identification of mode shapes that present nodal lines in the directions parallel to the boundaries of the plates. These eigen-modes are identified by a couple of integers (m,n) and by an angular frequency denoted by Ω_{mn}. Then, the approximated wave numbers (ξ_m,η_n) associated to the mode shape W_{mn} of the clamped plate given by formula (19) and the corresponding measured angular frequencies Ω_{mn}, are introduced in the inverse problem given by formula (12) and (13). These formula lead to the estimated elastic properties of the plate.

The reader will notice that the numerical effort necessary to perform this identification is very small: it only requires to firstly compute the matrices $[A]$ and $[B]$ appearing in expression (12), with the wave numbers ξ_m and η_n given by formula (19). The solution of system (12) is then performed in a least mean squares sense according to expression (13). Clearly, the main advantage of this identification method is to avoid the problem of explicitly solving the plate equations (1a,1b).

As already mentioned, this system will however be solved in the next sections only in order to provide a numerical validation of this identification method. In fact, the solutions thus obtained will play the role of the ''true'' displacement fields of the plate for given specially orthotropic plates with well-known elastic properties.

3.3. Numerical test-case

In this section, the method proposed for the evaluation of the elastic constants of a specially orthotropic plate submitted to both simply supported and clamped boundary conditions is analyzed through a numerical experiment. The vibration of the specially orthotropic plate is computed by solving the system (1a,1b) with the high-order finite differences method presented in details in [13]. The eigen-modes and eigen-frequencies coming from this direct problem, are then introduced in the inverse problem presented in the previous section. The structure is a 0.8×0.8 carbon/epoxy plate of thickness $h = 10^{-3}$ m, $E_1 = 120$ GPa, $E_2 = 10$ GPa, $G_{12} = 4.9$ GPa, $\nu_{12} = 0.3$, $\rho = 1510$ kg/m^3. This direct problem is solved on a 36×36 mesh-grid.

Tables 1 and 2 present the results of the identification of the elastic constants of the test-plate performed successively with the 10 first modes of the plate, from the 10th mode to

Table 1
Simply supported plate

	Elastic constants (GPa)		
	E_1	E_2	G_{12}
Computed constants	120	10	4.9
Identified constants with modes $1 \rightarrow 10$	119.99	9.998	4.901
Identified constants with modes $10 \rightarrow 20$	119.99	10	4.898
Identified constants with modes $1 \rightarrow 20$	119.99	10	4.898

Table 2
Clamped plate

	Elastic constants (GPa)		
	E_1	E_2	G_{12}
Computed constants	120	10	4.9
Identified constants with modes $1 \rightarrow 10$	115	9.4	3.6
Identified constants with modes $10 \rightarrow 20$	118	9.5	3.8
Identified constants with modes $1 \rightarrow 20$	125	9.5	3.8

the 20th mode, and finally with the 20 first modes of the plate. In Tables 1 and 2, the measured frequencies are the "true" frequencies given by the vibration calculus as already explained. More precisely, Table 1 refers to a simply supported plate, and Table 2 concerns a clamped plate.

The results of this numerical validation for the simply supported plate are "excellent" since the inverse problem (8), (12) and (13), solves exactly the system (1a,1b).

Fig. 1 shows the 9 first modes of the carbon/epoxy test-plate.

The results of the identification of the elastic constants of our test-plate from measurements on a clamped plate are less satisfactory than those obtained when the identification

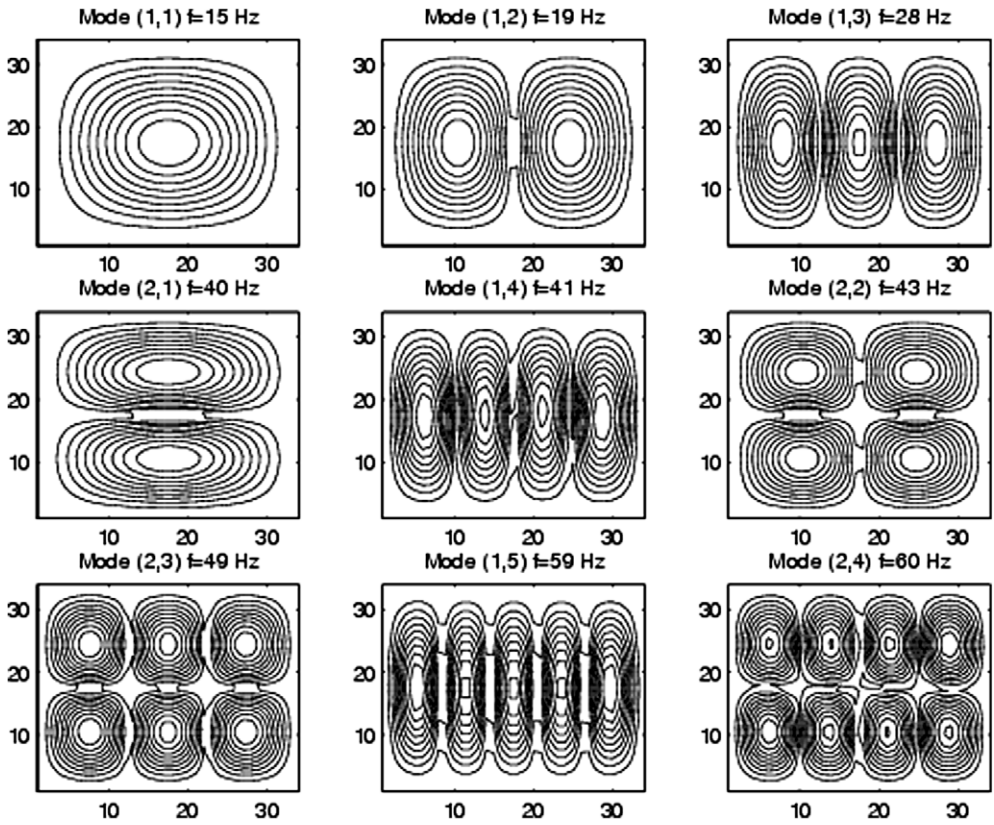

Fig. 1. Modes of the carbon/epoxy clamped test-plate.

is performed on the simply supported plate: the reason lies in formula (14) which only gives approximate values of the wave numbers of the clamped plate. Thus, the resulting wave numbers (ξ_n,η_n) given by (19) are also approximate values.

It was interesting to check the method in the case of disturbed measurements. That is the aim of Tables 3 and 4. A centered 10 Hz random noise has been added to the preceding computed frequencies, in order to model a measurement noise. More precisely, Table 3 presents the exact/disturbed frequencies, and Table 4 presents the elastic moduli identified in presence of this noise.

Table 4 shows that even with this 10 Hz noise which represents an error of 30% on the frequency of the first mode shape of the plate, the identification converges in a satisfactory way to the computed elastic moduli. In all cases, the in-plane shear modulus G_{12} is not described as well as the transverse moduli E_1 and E_2. The reason for this lies presumably in the fact that for the structure considered here, namely a carbon–epoxy plate, the shear modulus G_{12} is 24 times smaller than E_1, and 2 times smaller than

Table 3
Frequency noise

	Frequency (Hz)	
	Exact	Perturbed
Mode 1	15.39	10.97
Mode 2	19.65	18.18
Mode 3	28.54	31.67
Mode 4	40.42	35.52
Mode 5	41.97	38.36
Mode 6	43.38	40.41
Mode 7	49.67	46.66
Mode 8	59.57	60.61
Mode 9	60.24	57.96
Mode 10	75.41	72.40
Mode 11	78.41	73.56
Mode 12	85.90	88.37
Mode 13	95.10	94.55
Mode 14	119.09	123.41
Mode 15	123.51	123.17
Mode 16	129.11	128.30
Mode 17	131.44	134.90
Mode 18	142.87	143.12
Mode 19	153.27	150.30
Mode 20	167.61	169.33

Table 4
Clamped plate and perturbed frequencies

	Elastic constants (GPa)		
	E_1	E_2	G_{12}
Computed constants	120	10	4.9
Identified constants with modes $1 \rightarrow 10$	113	7.2	7.1
Identified constants with modes $10 \rightarrow 20$	123	10	3.5
Identified constants with modes $1 \rightarrow 20$	120	10	2.9

E_2, and therefore, its influence on the response of the structure remains small. In general, identification errors increase with parameters which have a small influence on the response of the system.

In conclusion, the identification of the elastic constants of a specially orthotropic plate presented in this section provides satisfactory results. The method is very simple to implement and fast to run. More precisely, computing an identification sequence requires to inverse a 3×3 positive definite matrix: we used a Cholesky method was used in the present article. This method works well at low frequency; i.e., on the first modes of the structure which are easy to measure with an impact testing hammer and an accelerometer, for example.

4. Middle frequency identification: The I.W.C. approach

4.1. Method principle

We first review the method developed in [11]. Based on rather high frequency assumptions, this approach leads to the assessment of a dispersion equation associated with the spatial vibrational field of a plate, denoted w. The main idea is to project the actual vibrational field on a set of inhomogeneous damped waves parameterized with the parameter triplets (k,θ,η), as follows:

$$o_{k,\theta,\eta}(x,y) = \mathrm{e}^{-\mathrm{i}k((1+\mathrm{i}\eta)x\cos\theta + y\sin\theta)} \quad \text{(damped plane waves)} \tag{20}$$

A correlation index between the harmonic field, namely the Fourier transform of the spatial field w, and each of the preceding parameterized waves o, called IWC (''inhomogeneous wave correlation''), depending on the wave propagation parameters is computed as follows:

$$\mathrm{IWC}(k,\theta,\eta) = \frac{\left|\int\int_S w o^{\star}_{k,\theta,\eta}\,\mathrm{d}x\,\mathrm{d}y\right|}{\sqrt{\left|\int\int_S |w|^2\,\mathrm{d}x\,\mathrm{d}y\right| \int\int_S |o_{k,\theta,\eta}|^2\,\mathrm{d}x\,\mathrm{d}y}} \tag{21}$$

where $a^{\star}$ denotes the complex conjugate of a. The reader will notice that the frequency does not appear explicitly in the preceding equations. As a consequence, this method can be classified as *a nonlinear correlation identification method* according to the terminology employed in the control theory literature.

In other words, $\mathrm{IWC}(k,\theta,\eta)$ represents the ''contribution'' of the wave o to the field w, i.e., the part of the energy supported by this wave of the total energy brought by the field. In the middle and high frequency domains, due to important modal overlapping phenomena, the actual field includes waves in all the directions θ. Therefore, the efficiency of this index strongly depends upon the modal overlap of the field w. The search for a maximum of this correlation index allows us to define a wave in any direction θ_i investigated, and thereby to assess the dispersion equation of the plate. Fig. 2 shows a 2D map of the displacements obtained at frequency 560 Hz and the wave numbers k estimated at this frequency with an angular step of $\Delta\theta = 5°$. $\hat{k}$ (respectively, ô) denotes the estimate of the wave number (respectively, the inhomogeneous wave) given by the maximization of the IWC index.

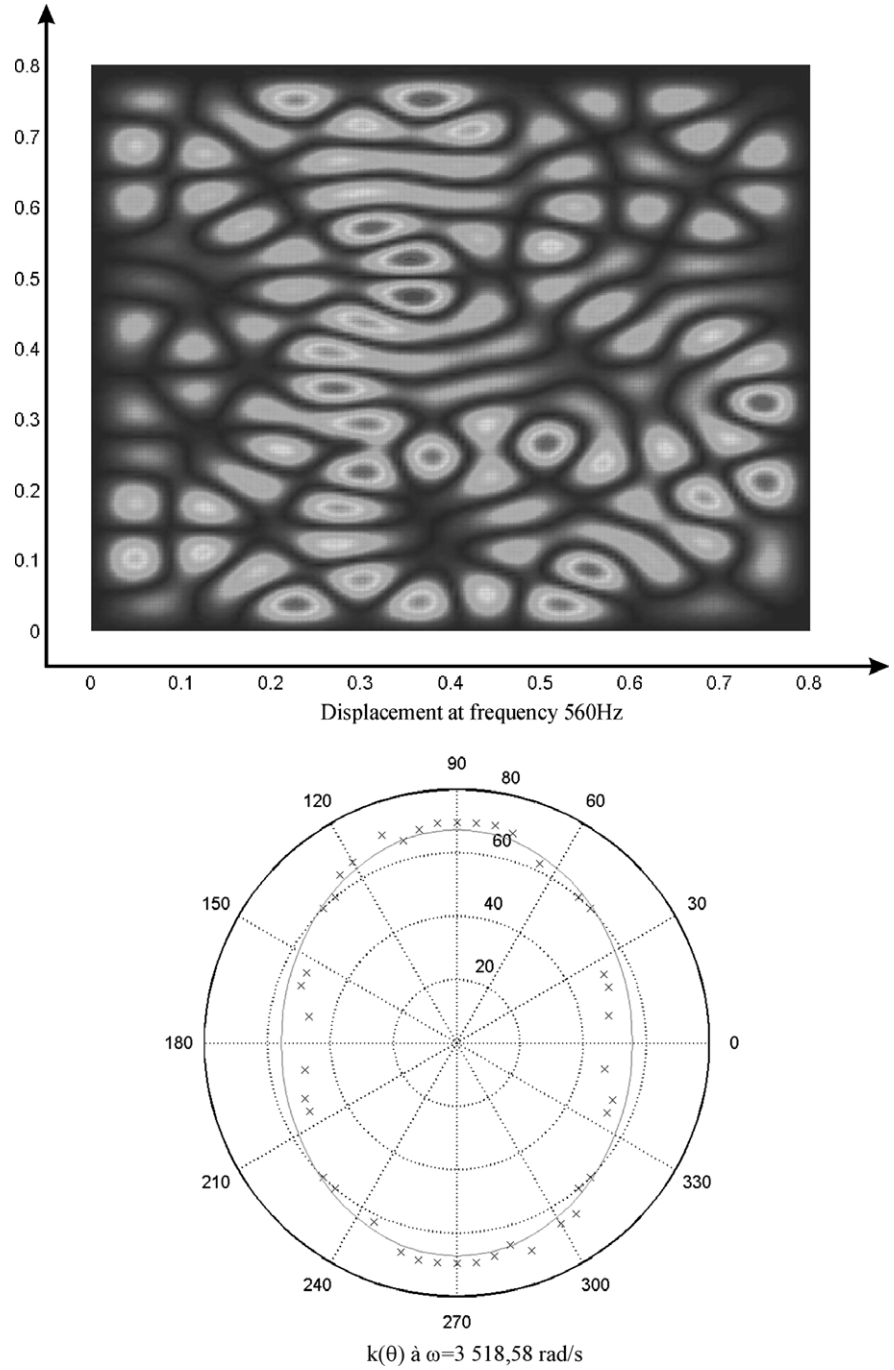

Fig. 2. Identification example at frequency $f = 560$ Hz. The angular step is $\Delta\theta = 5°$: displacement field (up), wavenumbers map (down).

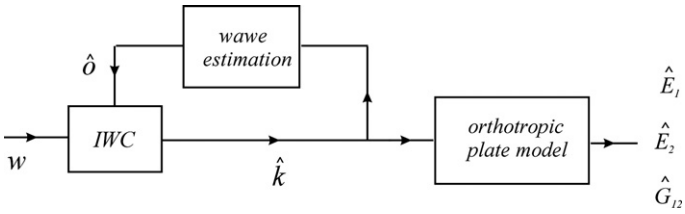

Fig. 3. Plate identification block diagram.

The IWC method implicitly permits to avoid the singular variations of the space field, thus making this approach somehow independent of the boundary conditions and of the signal source. This dispersion equation is then treated in order to provide a model of the equivalent orthotropic plate equivalent to the original one, as shown in Fig. 3.

In order to assess the flexural rigidities of the plate, i.e., (D_1,D_2,D_{12}), we refer to the classical formalism of [15]:

$$\sigma\omega^2 = k^4(D_1\cos^4\theta + D_2\sin^4\theta + D_{12}\cos^2\theta\sin^2\theta) \tag{22}$$

The estimates of the flexural rigidities $(\hat{D}_1, \hat{D}_2,$ and $\hat{D}_{12})$ of the equivalent orthotropic plate, come from the identification of the wave numbers $k(\theta)$ that maximize the IWC index in Eq. (21), and the minimization of a least squares index based on the formalism of Eq. (22). We therefore derive the mechanical constants, or more precisely their estimates, of the equivalent plate as follows:

$$\hat{E}_1 = \frac{12}{h^3}\hat{D}_1(1-\nu^2), \quad \hat{E}_2 = \frac{12}{h^3}\hat{D}_2(1-\nu^2), \quad \hat{G}_{12} = \frac{12}{h^3}\hat{D}_1(0,5\hat{D}_{12} - \nu\hat{D}_2) \tag{23}$$

4.2. Validation on a numerical experiment

The link between the direct problem and the inverse problem has been used in order to check the properties of the IWC method. The vibratory response of the special orthotropic carbon–epoxy plate, already used in Section 3.3, with dimensions $800 \times 800 \times 1$ mm excited by a white noise source term is computed according to the model presented in Section 2 above, leading to vibratory velocity (or displacement) maps lying on a 64×64 nodes mesh-grid.

We notice that the identification procedure presented above is limited in the frequency domain. The low frequency limitation comes from the constraint that one must work in a domain of strong modal density (remember that one single resonant mode only represents two points in the wave numbers plane while the estimation algorithm is trying to find it in any direction). Moreover, the high frequency limitation comes from the spatial sampling of the structure studied: Shannon's theorem states that it is impossible to identify the wave numbers greater than $f_s/2$ where f_s is the inverse of the spatial sampling step. The reader may notice that this plate presents an important structural orthotropy. The main material properties are (see Section 2 for notations): $E_1 = 120$ GPa, $E_2 = 10$ GPa, $G_{12} = 4.9$ GPa, $\nu_{12} = 0.3$ and $\rho = 1510$ kg m^{-3}.

Let us remember that the harmonic displacement maps are computed by solving system (1a,1b) in the frequency range [100–1800 Hz].

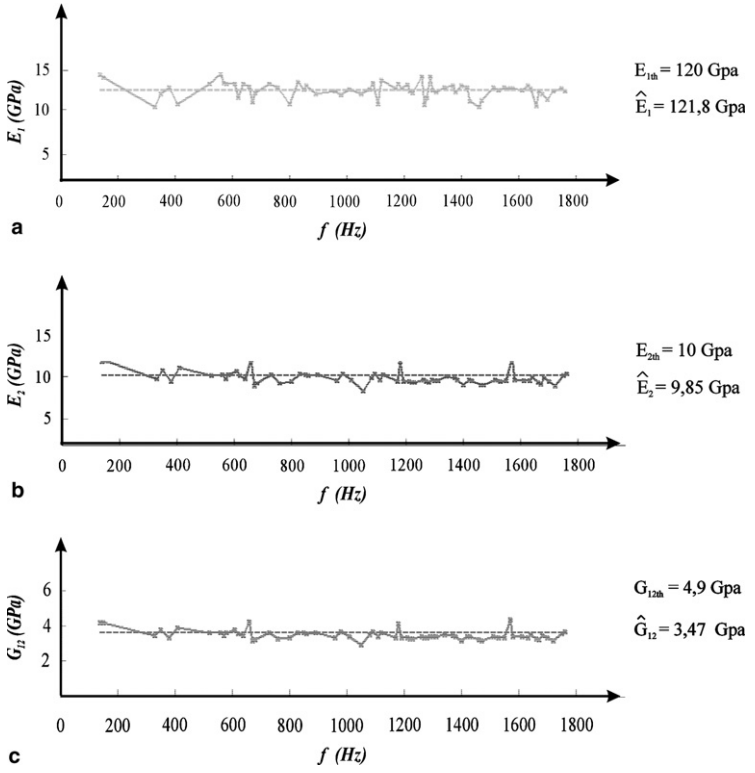

Fig. 4. Elastic constants identification from simulated measures (dot curves: mean values w.r.t. frequency).

Finally, the IWC method is applied at each measured frequency and the elasticity constants are obtained by an averaging procedure upon the intended frequency range, as shown in Fig. 4.

4.3. Comparison with the modal approach

The results obtained by both methods are rather satisfactory. In both cases, one can notice that the shearing factor is underestimated: the error is merely 20%. This can be explained by the fact that the rigidity constant G_{12} is very small, compared to the value of E_1, namely 4% of E_1. Since the identification is based on the global estimation of the three constants E_1, E_2 and G_{12}, the absolute errors made on each of these three constants are similar and comparatively, the relative error made on the estimation of G_{12} is greater than the one made on the other constants. Complete results are presented in Table 5.

Table 5
Comparison between modal approach and IWC approach: elastic constants (GPa)

	Elastic constants (GPa)		
	E_1	E_2	G_{12}
Computed constants	120	10	4.9
Modal approach	125	10	3.9
IWC approach	121.8	9.85	3.47

The main interest of the modal approach is its efficiency from the very first modes of the plate and especially in the low frequency domain where their measurements are easily available. Its main drawback is its great sensitivity to the boundary conditions.

The IWC method requires to work at rather high frequencies and needs a lot of data in order to accurately define the vibratory behavior of the structure. Its main advantage is to be almost independent of the boundary conditions and of the nature of the excitation source.

4.4. Application to an experimental case

Our concern is now to apply the preceding identification method to complex structures such as the ones encountered in the helicopter interior trim, in real excitation conditions in flight, as well as in artificial conditions. The harmonic vibratory field is measured by near-field acoustic holography (NAH) techniques. A bi-dimensional network of microphones provides experimental measures used to solve the reverse problem (back-propagation to the source plan phenomenon), i.e., to recover the field on the plane area so analyzed. The interested reader can refer to [16,17] for example. NAH permits the representation of the vibratory velocity fields with a convenient resolution. In fact, the measurement is done in the near-field of the source so the evanescent waves are taken into account in the reconstruction process of pressure and velocity spatial fields.

In practice, the study of stationary phenomena allows us not to measure simultaneously all the network nodes, so a reduced number of microphones have to be moved

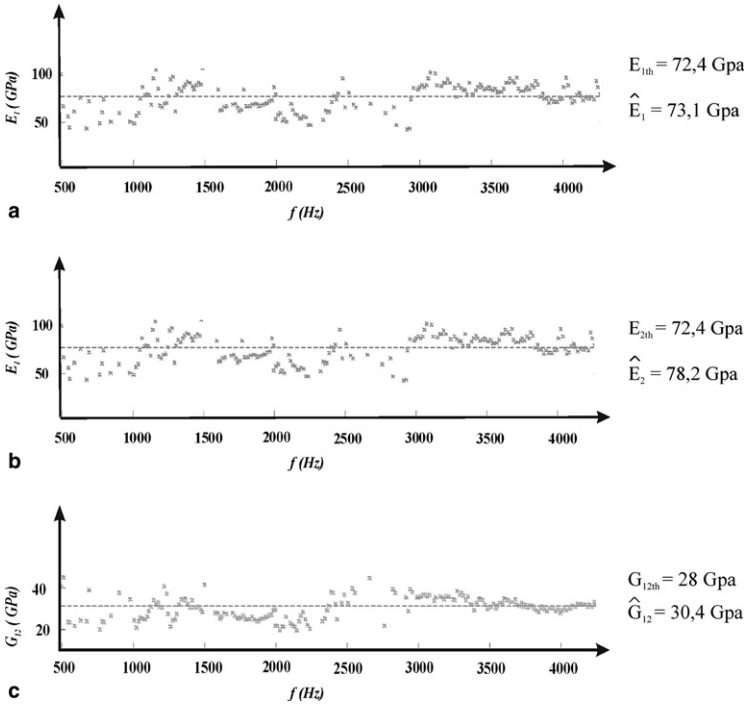

Fig. 5. Parameter identification from experimental measures (dot curves: mean values w.r.t. frequency).

to cover the source area, provided a phase reference is used during the whole measurement campaign (for more details, see for example [18]). Nevertheless, this method presents some limitations and requires rather sophisticated algorithms. Classically, filtering procedures are necessary to avoid problems linked to the amplification of noisy components during the back-propagation process. The reader interested in the algorithms we used, may refer to [19] for example.

In order to justify using acoustic holography techniques in the identification procedure, an experimental setup has been realized, using an homogenous and isotropic aluminium plate with dimensions $800 \times 800 \times 5$ mm. The mechanical properties of this plate are known "a priori": $E_1 = E_2 = 72.4$ GPa, $G_{12} = 28$ GPA, $v_{12} = 0.3$ and $\rho = 2770$ kg m^{-3}. This plate is excited by a modal shaker providing a white noise acceleration in the frequency range of interest [10;6000 Hz]. The excitation signals are captured by an accelerometer located at the excitation point and the acoustic pressure is measured by an antenna with 64 microphones. A fixed microphone acting as a phase reference, allows us to move the antenna in order to entirely cover the plate surface, thus defining a final network with 40×40 nodes leading to the vibration velocity field w of the IWC approach. Complete identification results are given in Fig. 5.

The dispersion of the results thus obtained is greater than in simulation (see Fig. 5). Nevertheless, they remain quite satisfactory since we consider their mean values with respect to the frequency, and this, despite the difficulties of the measurement procedure and the errors due to the holography method itself.

5. Conclusion

In this paper, we presented and discussed two identification methods of the plate elasticity constants based, respectively, on a modal representation and on an ondulatory description of the vibrations. These methods have been validated and compared on a numerical example and an experimental example. Our work shows that we have now at our disposal a complete range of identification tools covering the low frequency domain as well as the high frequency domain. We can now apply these techniques to more complicated structures such as complex panels made of composite materials together with stiffeners. Indeed, these structures will be modelled by plane orthotropic descriptions whose vibratory behavior is predictable under given excitations.

Moreover, these methods allow us to simplify the numerical modelling of complex structures (homogenization problem): finite element representations, statistical energy analysis, ... so as to derive their vibro-acoustic behavior.

As an application, it will be interesting to model complex structures such as an helicopter cabin, in order to derive the inner acoustic radiation and to better control its propagation.

In a forthcoming work, we will investigate the delicate problem of the structural damping estimation which is not yet satisfactorily treated, neither by modal nor correlation approaches.

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